Resolution Limits of Non-Adaptive 20 Questions Search for Multiple Targets
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Abstract

We study the problem of simultaneous search for multiple targets over a multidimensional unit cube and derive the fundamental resolution limit of non-adaptive querying procedures using the 20 questions estimation framework. The performance criterion that we consider is the achievable resolution, which is defined as the maximal \( L_\infty \) norm between the location vector and its estimated version where the maximization is over the possible location vectors of all targets. The fundamental resolution limit is then defined as the minimal achievable resolution of any non-adaptive query procedure. We derive non-asymptotic and second-order asymptotic bounds on the minimal achievable resolution by relating the current problem to a data transmission problem over a multiple access channel, using the information spectrum method by Han and borrowing results from finite blocklength information theory for random access channel coding. Our results extend the purely first-order asymptotic analyses of Kaspi et al. (ISIT 2015) for the one-dimensional case. Specifically, we consider more general channels, derive the non-asymptotic and second-order asymptotic results and establish a phase transition phenomenon.

Index Terms
Second-order asymptotics, information spectrum method, multiple access, random access, measurement-dependent noise

I. INTRODUCTION

Rényi initiated the study of a random search problem in [1], which was motivated by diverse applications such as medical diagnosis, chemical analysis and searching for a root of an equation. The problem was later known as the 20 questions game for estimation. In this problem, there are two players: an oracle and a questioner. The oracle knows the realization of a target random variable, say \( S \in [0, 1] \). The goal of the questioner is to accurately estimate \( S \) by posing as few queries as possible. In [1], the oracle’s responses are assumed to be noisy versions of the true answers to the queries posed by the questioner. In particular, the behavior of the oracle is modeled by a memoryless noisy channel and the noisy responses are the outputs of this channel with true answers being the inputs. Of central interest is the determination of the minimum number of queries needed to accurately estimate the target random variable using a particular performance criterion.

Jedynak et al. [2] studied this problem when the criterion is the entropy of the posterior distribution of the target random variable, which is to be minimized. Specifically, Jedynak proposed an optimal non-adaptive query procedure named the dyadic policy and an optimal adaptive query procedure based on the probabilistic bisection policy in [3]. An non-adaptive query procedure asks a fixed number of predetermined queries while an adaptive query procedure can pose a random number of queries and each query can depend previous queries and the received noisy responses. Readers can refer to [4, Fig. 1] for more details. Subsequently, the results of Jedynak were generalized to a collaborative case [5], a decentralized case [6] and a multiple target case [7].

Inspired by the future research directions pointed out in [2], a sequence of recent works including [4], [8]–[10] considered the more natural performance criterion such as the the \( L_2 \) norm or the \( L_\infty \) norm of the estimation error; furthermore, a subset of these works, such as [4], [9] considered query-dependent or measurement-dependent noise where the noisy channel that models the behavior of the oracle can depend on the query. Such a measurement-dependent model is applicable to target localization tasks, e.g., in a sensor network where the noise can accumulate when collecting responses or in a human-in-the-loop estimation model where workers make errors with error probability that is a function of the query.

Most previous work focuses on the case of a single target with rare exceptions, e.g., [7], [11]. As discussed in the pioneering work on multiple target search by Rajan et al. [7], the problem of simultaneous search for multiple targets is motivated by diverse applications, ranging from localizing faces in pictures to finding quasars in astronomical data. To advance the understanding of fundamental limits of multiple target search, in this paper, we take a finite blocklength information theoretical look at the problem. Our main contributions and the comparison of our results with the state-of-the-arts are as follows.

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A. Main Contributions

We obtain bounds on performance of 20 questions non-adaptive query procedures for the problem of simultaneous search for multiple targets over the unit cube with a measurement-dependent noise model (cf. Definition 1). Our performance criterion is the $L_\infty$ norm. Under our problem formulation, we define the fundamental resolution limit as the minimal achievable resolution $\delta^*(n, k, d, \varepsilon)$ (cf. Eq. (5)) of any non-adaptive query procedure. This limit is a function of the number of targets $k$, the dimension of the search space $d$, the number of queries $n$ and a maximum tolerable excess-resolution probability $\varepsilon$.

Our main results include non-asymptotic bounds (cf. Theorems 1 and 2) and a second-order asymptotic bound (cf. Theorem 3) on the fundamental limit $\delta^*(n, k, d, \varepsilon)$. In particular, our second-order result in Theorem 3 provides approximations to the performance of any optimal non-adaptive query procedure with finitely many queries. These approximations are validated via numerical examples. In order to prove our results, we relate the current problem to a data transmission problem over a multiple access channel, employ a random coding idea, apply the information spectrum method in [12] and use inequalities from random channel coding [13] in addition to results from finite blocklength information theoretical tools of [13]–[15].

As a corollary of our result, we establish a phase transition phenomenon of the minimal excess-resolution probability as a function of the resolution decay rate. In particular, when the number of queries increases, the excess-resolution probability of an optimal non-adaptive query procedure switches rapidly from zero to one as a function of the resolution decay rate, where the critical threshold is a function of the capacity of the measurement-dependent channel (cf. the first remark of Theorem 3 and Figure 1). Furthermore, we show that our results hold not only for a fixed number of targets but also for the case where the number of targets increases with the number of queries at a certain rate. Finally, our results extend easily to the $L_2$ norm performance criterion, which is also bounded by the fundamental limit $\delta^*(n, k, d, \varepsilon)$.

B. Comparisons with Existing Results

Among the vast literature on 20 questions, very few are on simultaneous search for multiple targets, e.g., [7], [11]. The authors of [7] considered the minimal posterior entropy criterion introduced in [2] and is thus not comparable to our work. Therefore, we mainly compare our results with [11, Theorem 1]. The first difference is that we consider any measurement-dependent channel with transition matrix $P_{X|S}$ and passes these answers through a measurement-dependent channel (BSC) for $d = 1$. Secondly, we derive the second-order asymptotic result on the fundamental limit of non-adaptive query procedures which provide benchmarks for practical settings with finitely many queries while [11] only performed a first-order asymptotic analysis characterizing the resolution decay rate in the limit of an infinite number of queries. Furthermore, we offer new insights beyond those of [11], including an interesting phase transition phenomenon that demonstrates a sharp degradation of the excess-resolution probability in the noise variance as a function of the resolution decay rate. Our proof techniques significantly differ from [11] where asymptotic Shannon techniques including the Fano’s inequality were used, while we apply the recently developed finite blocklength information theoretical tools of [13]–[15].

II. Problem Formulation

Notation

Random variables and their realizations are denoted by upper case variables (e.g., $X$) and lower case variables (e.g., $x$), respectively. All sets are denoted in calligraphic font (e.g., $\mathcal{X}$). Let $X^n := (X_1, \ldots, X_n)$ be a random vector of length $n$. We use $\Phi^{-1}()$ to denote the inverse of the cumulative distribution function (cdf) of the standard Gaussian. We use $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ to denote the sets of real numbers, positive real numbers and integers respectively. Given any two integers $(m, n) \in \mathbb{N}^2$, we use $[m : n]$ to denote the set of integers $\{m, m+1, \ldots, n\}$ and use $[m]$ to denote $\{1 : m\}$. Given any $(m, n) \in \mathbb{N}^2$, for any length $m$ vector $a = \{a_1, \ldots, a_m\}$, the infinity norm is defined as $||a||_\infty := \max_{i \in [m]} |a_i|$ and the $L_2$ norm is defined as $||a||_2 := \sqrt{\sum_{i \in [n]} a_i^2}$. The set of all probability distributions on a finite set $\mathcal{X}$ is denoted as $\mathcal{P}(\mathcal{X})$ and the set of all conditional probability distributions from $\mathcal{X}$ to $\mathcal{Y}$ is denoted as $\mathcal{P}(\mathcal{Y}|\mathcal{X})$. Furthermore, we use $\mathcal{F}(\mathcal{S})$ to denote the set of all probability density functions on a set $\mathcal{S}$. All logarithms are base $e$ unless otherwise noted. Finally, we use $\mathbb{I}()$ to denote the indicator function.

A. Problem Formulation under the Noisy 20 Questions Framework

Given finite integers $(k, d) \in \mathbb{N}^2$, let $\mathbf{S}^k := (\mathbf{S}_1, \ldots, \mathbf{S}_k)$ be a sequence of $k$ target random vectors, where for each $i \in [k]$, the random vector $\mathbf{S}_i = (S_{i,1}, \ldots, S_{i,d})$ is generated independently from an arbitrary pdf $f_{S_i}$ on the unit cube $[0,1]^d$. Note that any search problem over a bounded $d$-dimensional region is equivalent to a search problem on the unit cube of dimension $d$.

In the problem of simultaneous search for multiple targets, a player aims to accurately estimate the random location vectors $\mathbf{S}^k$ of all $k$ targets by posing a sequence of queries $\mathcal{A}^n = \{A_1, \ldots, A_n\} \subseteq [0,1]^n$ to an oracle knowing $\mathbf{S}^k$. After receiving the queries, the oracle finds binary answers $\{X_i = 1 \iff \exists j \in [k] : (S_{j,1}, \ldots, S_{j,d})\} \in [n]$ and passes these answers through a measurement-dependent channel with transition matrix $P_{Y^n|X^n} \in \mathcal{P}(\mathcal{Y}^n|\{0,1\}^n)$ yielding noisy responses $Y^n = (Y_1, \ldots, Y_n)$. Given the
Given any other with respect to the target resolution $S$ where $\nu$ constraint in (2) is satisfied for the measurement-dependent BSC. For this case, a valid choice of the function $f$ as the input with probability $\zeta = 1 \cdot \nu$ (3), similarly to [4], we assume that the measurement-dependent channel $P^A_{Y|X}$ depends on the query $A$ only through its size, i.e., $P^A_{Y|X}$ is equivalent to a channel with state $P^q_{Y|X}$ where the state $q$ is a function of $|A|$. Specifically, $q = f(|A|)$ where the function $f : [0, 1] \rightarrow \mathbb{R}_+$ is a bounded Lipschitz continuous function with parameter $\mu$, i.e., $|f(q_1) - f(q_2)| \leq \mu|q_1 - q_2|$ and $\max_{q \in [0,1]} f(q) < \infty$. A simple choice of $f(\cdot)$ is the identify function, i.e., $f(|A|) = |A|$ and for this case $\mu = 1$. Furthermore, if $f(\cdot)$ is a constant value function, then the channel reduces to a measurement-independent channel. Thus, the channel model above unifies both measurement-independent and measurement-dependent channels.

For any $q \in [0, 1]$ and any $\xi \in (0, \min(q, 1 - q))$, similarly to [4], we assume that the measurement-dependent channel is continuous in the sense that there exists a constant $c(q)$ depending on $q$ only such that
\[
\max \left\{ \left\| \log \frac{P^q_{Y|X}}{P^{q+\xi}_{Y|X}} \right\|_\infty, \left\| \log \frac{P^q_{Y|X}}{P^{q-\xi}_{Y|X}} \right\|_\infty \right\} \leq c(q)\xi. \tag{2}
\]

One example of the measurement-dependent channel is as follows.

**Definition 1.** Given any $A \subseteq [0, 1]$, a channel $P^A_{Y|X}$ is said to be a measurement-dependent Binary Symmetric Channel (BSC) with parameter $\zeta \in (0, 1]$ if $X = Y = \{-1, 1\}$ and
\[
P^A_{Y|X}(y|x) = (\zeta f(|A|))^{1(y\neq x)}(1 - \zeta f(|A|))^{1(y=x)}, \quad \forall \ (x, y) \in \{-1, 1\}^2. \tag{3}
\]

This definition generalizes [10, Theorem 1], where the authors considered a measurement-dependent BSC with parameter $\zeta = 1$ and the function $f(|A|) = |A|$. Note that for a measurement-dependent BSC with parameter $\zeta$, the output bit is the same as the input with probability $1 - \zeta f(|A|)$ and the output bit flips the input with probability $\zeta f(|A|)$. It can be verified that the constraint in (2) is satisfied for the measurement-dependent BSC. For this case, a valid choice of the function $f(\cdot)$ should satisfy $\nu f(|A|) \leq 1$ for any $A \subseteq [0, 1]$. In particular, we only consider Lipschitz continuous functions $f(\cdot)$ so that $\zeta f(|A|) \leq 1$ for any $A \subseteq [0, 1]$. This is because having a crossover probability of greater than $\frac{1}{2}$ is impractical for a BSC.

**C. Definition of the Fundamental Limit**

A non-adaptive query procedure for simultaneous search for multiple targets is defined as follows.

**Definition 2.** Given any $(n, k, \delta, \varepsilon) \in \mathbb{N}^3$, $\delta \in \mathbb{R}_+$ and $\varepsilon \in [0, 1)$, an $(n, k, \delta, \varepsilon)$-non-adaptive query procedure for noisy targets consists of

- $n$ queries $(A_1, \ldots, A_n)$ where each $A_i \subseteq [0, 1]^d$.
- and a decoder $g : Y^n \rightarrow S_m \subseteq [0, 1]^{kd}$

such that the excess-resolution probability satisfies
\[
P^A_{\varepsilon}(n, k, \delta, \varepsilon) := \sup_{f_\delta \in \mathcal{F}(n)} \max \left\{ \Pr \left\{ \exists i \in [k] : \min_{\hat{S} \in S_m} \left| \hat{S} - S_i \right|_{\infty} > \delta \right\}, \Pr \left\{ \exists \hat{S} \in S_m : \min_{i \in [k]} \left| \hat{S} - S_i \right|_{\infty} > \delta \right\} \right\} \leq \varepsilon, \tag{4}
\]

where $S_m = \{ \hat{S}_1, \ldots, \hat{S}_{|S_m|} \}$ is the list of estimated location vectors output by the decoder $g$ using noisy responses $Y^n$ and $\hat{S}_j$ denotes the $j$-th element of the vector $\hat{S}$ in $S_m$.

Note that the the list size $|S_m|$ could be smaller than the number of targets $k$ when two or more targets are close to each other with respect to the target resolution $\delta$. 

We remark that the first probability term ensures that for every target location vector \( S \), we can find a vector \( \hat{S} \) in the estimated set \( S_m \) such that the estimate is accurate within resolution \( \delta \) and the second probability term ensures that the size of \( S_m \) is minimized in the sense that every vector \( \hat{S} \) in \( S_m \) lies in the proximity of at least one target location vector \( S_i \).

The minimal achievable resolution of an optimal non-adaptive query procedure is then defined as

\[
\delta^*(n, k, d, \varepsilon) := \inf \{ \delta : \exists \text{ an } (n, k, d, \delta, \varepsilon)-\text{non-adaptive query procedure} \}.
\]

(5)

We remark that the minimal achievable sample complexity \( n^*(\delta, k, d, \varepsilon) \) can be defined analogously to (5) and it can be expressed as a function of \( \delta^*(n, k, d, \varepsilon) \). Thus, we focus on the characterization of \( \delta^*(n, k, d, \varepsilon) \) in this paper.

III. MAIN RESULTS AND DISCUSSIONS

A. Preliminary Definitions

To present our results and proposed query procedures, the following definitions are needed. For simplicity, given any \( k \in \mathbb{N} \) and \( J \subseteq [k] \), we use \( X_J \) to denote the collection of random variables \( \{X_j\}_{j \in J} \) and use \( X_J^n \) to denote the collection of sequences \( \{X^n(j)\}_{j \in J} \). We use \( x_J \) and \( x^n_J \) similarly for particular realizations. Given any \( p \in [0, 1] \) and \((x^k, y) \in [0, 1]^k \times \mathcal{Y} \), define the joint distribution

\[
P_{X^n \mid Y \mid Z}^{f(p), k}(x^k, y) := \prod_{i \in [k]} P_{X}^{f(p)}(x_i) P_{Y \mid Z}^{f(p)}(y \mid \exists i \in [k] : x_j = 1),
\]

(6)

where \( P_X \) denotes a Bernoulli distribution with parameter \( p \) and \( P_{Y \mid Z}^{f(p)} \) denotes a measurement-dependent channel. Note that the induced noisy channel (conditional distribution) \( P_{Y \mid X}^{f(p), k} \) is a binary input multiple access channel (MAC) that satisfies the permutation-invariance, reducibility, friendliness and interference assumptions in [13].

Furthermore, for any \( \gamma > 0 \), define the following sets of sequences

\[
\mathcal{D}^{n,t}_{J} = \{(x^n_{|J|}, y^n) \in [0, 1]^n \times \mathcal{Y}^n : \sum_{t \in [n]} t^{f(p), t}_{J}(x_{t \mid |J|}; y) > d(t - |J|) \log M + \gamma \}, J \subseteq [t],
\]

(10)

\[
\mathcal{D}^{n,t}(\gamma) := \bigcap_{J \subseteq [t]} \mathcal{D}^{n,t}_{J}(\gamma).
\]

(11)

Finally, for any \((k, M) \in \mathbb{N}^2\), let \( \mathcal{L}(k, M) \) be the set of all length-\( k \) vectors whose elements are ordered in increasing order and each element takes values in \([M]\), i.e.,

\[
\mathcal{L}(k, M) := \{(i_1, \ldots, i_k) \in [M]^k : \forall (j, l) \in [k]^2 \text{ s.t. } j < l, i_j < i_l\}.
\]

(12)

B. Non-Asymptotic Bounds

In this section, we provide non-asymptotic bounds on the excess-resolution probability which hold for non-adaptive query procedures with any number of queries. Recall the definition of the joint distribution \( P_{X^n \mid Y}^{f(p), k} \) in (6).

Our first result characterizes the performance of the non-adaptive query procedure in Algorithm 1 with finitely many queries.

Theorem 1. Given finite integers \((n, k, d) \in \mathbb{N}^3\), for any \((M, p, \eta, \gamma) \in \mathbb{N} \times (0, 1) \times \mathbb{R}_+^2\), there exists an \((n, k, d, 1/M^d, \varepsilon)\)-non-adaptive query procedure where

\[
\varepsilon \leq 4n \exp(-2M^d \eta) + \exp(n \mu \psi(f(p))) \left( (k + 1)2^k \exp(-\gamma) + \max_{t \in [n]} \Pr_{X^n_{|J|}^{f(p), k}} \left( X^n_{|J|} \neq Y^n \right) \right). \]

(13)

\(^1\)This definition of the excess-resolution probability in (4) results from discussions with anonymous reviewers for our previous paper [4].
Algorithm 1 Non-adaptive query procedure for simultaneous search for multiple targets

Input: The number of queries \( n \in \mathbb{N} \) and three parameters \( (M, p, \gamma) \in \mathbb{N} \times (0, 1) \times \mathbb{R}_+ \)

Output: A set of estimated location vectors of the targets \( S_m \subseteq [0,1]^{kd} \)

Partition the unit cube of dimension \( d \) into \( M^d \) equal-sized disjoint cubes \( \{S_1, \ldots, S_{M^d}\} \).
Generate \( M^d \) binary vectors \( \{x^n(1), \ldots, x^n(M^d)\} \) where for each \( i \in [M^d] \), the binary vector \( x^n(i) \) is generated i.i.d. from a Bernoulli distribution with parameter \( p \).

\[ l \leftarrow 1. \]

while \( l \leq n \) do
    Pose the \( l \)-th query to the oracle as whether there are targets in the region \( A_l \) where
    \[ A_l := \bigcup_{i \in [M^d]; x_l(i) = 1} S_i \]
    Obtain a noisy response \( y_l \) from the oracle.
    \[ l \leftarrow l + 1. \]

end while

Initialize \( S_m \) as the empty set.
\[ t \leftarrow k. \]

while \( t > 0 \) do
    if \( \exists \) a tuple \( (i_1, \ldots, i_t) \in \mathcal{L}(t, M^d) \) such that \( (X^n(i_1), \ldots, X^n(i_t), Y^n) \in \mathcal{D}^{n,t}(\gamma) \) (cf. (11)) then
        Return \( S_m \) as the centers of cubes \( S_{i_j} \) with \( j \in [t] \)
        \[ t \leftarrow 0. \]
    else
        \[ t \leftarrow t - 1. \]
    end if
end while

The proof of Theorem 1 combines the random coding idea, the information spectrum method for the multiple access channel [18] and the change-of-measure technique [19] and is provided in Section IV.

We remark that the first term \( 4n \exp(-2M^d \eta) \) results from the atypicality of the measurement-dependent channel compared with the measurement-independent channel \( P_{Y|X}^{f(p),t} \) induced by the joint distribution \( P_{X,Y}^{f(p),t} \). The multiplicative term \( \exp(n\mu_p(c(f(p)))) \) results from the change-of-measure technique, the assumption on the channel in (2) and the assumption on the Lipschitz continuous function \( f(\cdot) \). Similar terms also appear in our previous result on searching for a single multidimensional target [4, Theorem 1].

The terms inside the bracket result from the information spectrum method for a MAC. In particular, the maximum over \( t \in [k] \) is taken to control the worst case excess-resolution probability. This is because the number of distinct quantized versions of the location vectors after the quantization operation in Algorithm 1 is a random number that takes values in \([k]\) and depends on the target location vectors \( S^k = (S_1, \ldots, S_k) \in [0,1]^{kd} \).

We then present a converse result, which lower bounds the resolution of any non-adaptive query procedure subject to a constraint on the excess-resolution probability.

**Theorem 2.** Given any \( (n, k, d, \varepsilon) \in \mathbb{N}^3 \times (0, 1) \), for any \( \beta \in (0, \frac{\varepsilon}{2}] \) and \( \gamma \in \mathbb{R}_+ \) such that \( 2^k \exp(-\gamma) + 2k^2 d \beta \in (0, 1 - \varepsilon) \), the minimal achievable resolution \( \delta^*(n, k, d, \varepsilon) \) satisfies

\[-\log \delta^*(n, k, d, \varepsilon) \leq \sup_{A^n \in [0,1]^d} \left\{ \psi \in \mathbb{R}_+: \max_{t \in [k]} \Pr_{X^n|Y^n} \left\{ \sum_{t \in [n]} |A_t|^t \psi(X_t, |X^n_y|) \leq dt \psi + dt \log \beta - \gamma \right\} \right. \]

\[ \leq \varepsilon + 2^k \exp(-\gamma) + 2k^2 d \beta, \quad (14) \]

where the joint distribution \( P_{X^n|Y^n}^{A^n} \) is given by \( P_{X^n|Y^n}^{A^n} = \prod_{t \in [n]} P_{X^n|Y^n}^{f(|A^t|, k)}(x_t, y_t) \), i.e.,

\[ P_{X^n|Y^n}^{A^n}(x^n, y^n) = \prod_{t \in [n]} \left( \prod_{i \in [k]} P_X^{A_t}(x_i,t) \right) P_{Y|X}^{f(|A^t|)}(y_t | \exists i \in [k]: x_i,t = 1) \]

(15)

with \( P_X^{A_t} \) being the Bernoulli distribution such that \( P_X^{A_t}(1) = |A_t| \).

The proof of Theorem 2 is provided in Section V. To prove Theorem 2, we first lower bound the excess-resolution probability of an non-adaptive query procedure with the error probability of estimating quantized versions of the target location vectors that
are independently and uniformly distributed over the unit cube. Subsequently, we show that the latter problem is closely related with transmitting independent messages over a codebook-dependent multiple access channel. Finally, using the information spectrum method for the MAC by Han [18], we obtain the desired lower bound on the achievable resolution of an optimal non-adaptive query procedure.

We remark that the non-asymptotic bound in Theorem 2 is difficult to calculate, especially for relatively large number of queries $n$. This is because a supremum over all possible queries $\mathcal{A}^n \in [0,1]^n$ is involved. However, as demonstrated in the proof of Theorem 3, for $n$ sufficiently large, the bound in Theorem 2 can be approximated with simple equations independent of the queries $\mathcal{A}^n$, and such approximations are collectively known as second-order asymptotics [14].

C. Second-Order Asymptotic Approximation and Further Discussions

Define the capacity of the above binary-input MAC as

$$C(k) := \max_{p \in (0,1)} C_0(p,k).$$  (16)

Let $\mathcal{P}_k$ be the set of optimizers $p^*$ for the optimization problem in (16). The dispersion for the channel is then defined as

$$V(k,\varepsilon) := \left\{ \begin{array}{ll} \min_{p^* \in \mathcal{P}_k} V_0(p,k) & \text{if } \varepsilon \leq 0.5, \\ \max_{p^* \in \mathcal{P}_k} V_0(p,k) & \text{if } \varepsilon > 0.5. \end{array} \right.$$  (17)

Since we consider finite input and output alphabets, similar to [15, Lemma 47], the dispersion $V(k,\varepsilon)$ is finite.

**Theorem 3.** For any finite numbers $(k,d) \in \mathbb{N}^2$ and any $\varepsilon \in (0,1)$, the achievable resolution $\delta^*(n,k,d,\varepsilon)$ of an optimal non-adaptive query procedure satisfies

$$-\log \delta^*(n,k,d,\varepsilon) = \frac{nC(k) + \sqrt{nV(k,\varepsilon)\Phi^{-1}(\varepsilon)}}{dk} + O(\log n),$$  (18)

where the $O(\log n)$ term is lower bounded by $-\frac{\log n}{2dk}$ and upper bounded by $\log n + O(1)$ and the $O(1)$ term in the upper (converse) bound depends on the number of targets $k$ and the dimension $d$.

The proof of Theorem 3 is provided in Section VI. In particular, we apply the Berry-Esseen theorem [20], [21] to our non-asymptotic bounds and use the inequalities in [13, Lemmas 1 and 2]. In our achievability proof of Theorem 3, we use the non-adaptive query procedure in Algorithm 1 and thus prove its second-order asymptotic optimality.

We make several remarks. If we let $\varepsilon^*(n,k,d,\delta)$ be the minimal excess-resolution probability of any non-adaptive query procedure, then Theorem 3 implies that for $n$ sufficiently large

$$\varepsilon^*(n,k,d,\delta) = \Phi\left(\frac{-dk \log \delta - nC(k)}{\sqrt{nV(k,\varepsilon)}}\right) + o(1).$$  (19)

Note that (19) implies a phase transition phenomenon. Specifically, when the target resolution decay rate $-\frac{\log \delta}{n}$ is strictly greater than $\frac{C(k)}{dk}$, then the excess-resolution probability tends to one as the number of queries $n$ tends to infinity; on the other hand, when the target resolution decay rate is strictly less than the critical rate $\frac{C(k)}{dk}$, then the excess-resolution probability vanishes as the number of queries $n$ increases. See Figure 1 for an illustration of the phase transition phenomenon for a measurement-dependent BSC.

Furthermore, we comment on the difficulties in the proof induced by the multiple targets. If one naively applies the Berry-Esseen theorem to our non-asymptotic bounds, then unmatched asymptotic results are obtained, i.e.,

$$\max_{p \in (0,1)} \min_{t \in [k]} \frac{C_0(p,t)}{dt} \leq \lim_{n \to \infty} \frac{-\log \delta^*(n,k,d,\varepsilon)}{n} \leq \min_{t \in [k]} \max_{p \in (0,1)} \frac{C_0(p,t)}{dt}. $$  (20)

To obtain tight results, we need to find a saddle point for the above maximin and minimax problems. However, in general, verifying the existence of such a saddle point for every measurement-dependent channel is not easy. To solve this problem, we find that the inequality in [13, Lemmas 1], which was derived for random channel coding, is helpful. Since the MAC channel induced by the joint distribution in (6) satisfies the conditions in [13, Lemma 1], we then have $\min_{t \in [k]} \frac{C_0(p,t)}{dt} = \frac{C_0(p,k)}{dt}$. In the converse part, by applying [13, Lemma 1], we can avoid using minimax bounds to upper bound a maximin term and verify that the maximin term $\max_{p \in (0,1)} \min_{t \in [k]} \frac{C_0(p,t)}{dt}$ is also the dominant term in the inverse (upper) bound. The details are quite involved and given in Section VI-B.

Another remark is that Theorem 3 implies a strong converse of the resolution decay rate of an optimal non-adaptive query procedure, i.e., for any $\varepsilon \in (0,1)$,

$$\lim_{n \to \infty} \frac{-\log \delta^*(n,k,d,\varepsilon)}{n} = \frac{C(k)}{dk} = \max_{p \in (0,1)} \min_{t \in [k]} \frac{C_0(p,t)}{dt}. $$  (21)
Algorithm 1 is plotted and compared to the theoretical predictions in Theorem 3.

The performance of Algorithm 1 under various choices of the Lipschitz continuous functions can be second-order asymptotically optimal by our converse proof of Theorem 3. The minimization over $t \in [k]$ in (21) follows from the fact that given a certain resolution, the number of cubes containing targets might be fewer than the total number of targets. This is because in a query procedure, we need to do a partition of the unit cube into equal-sized disjoint regions and quantize the targets $s^k = (s_1, \ldots, s_k)$. Two targets $(s_i, s_j)$ might be quantized into the same cube if they are too close with respect to a given resolution $\delta$. To ensure that our result holds for all possible cases, a minimization over the number of quantized targets accounts for the worst case. Furthermore, to maximize the performance of the non-adaptive query procedure, we choose the best possible codebook by maximizing over the parameter $p \in (0, 1)$. Such intuition is confirmed to be second-order asymptotically optimal by our converse proof of Theorem 3.

Furthermore, if one uses the $L_2$ norm instead of the $L_\infty$ norm as the performance criterion in the definition of the excess-resolution probability in (4), we can define the corresponding fundamental limit as $\delta^*_{L_2}(n, k, d, \varepsilon)$. We then have $\delta^*_\infty(n, k, d, \varepsilon) \leq \delta^*_{L_2}(n, k, d, \varepsilon) \leq \sqrt{d} \delta^*(n, k, d, \varepsilon)$ and thus the second-order asymptotic result in Theorem 3 also holds under the $L_2$ norm performance criterion when the dimension $d$ is finite. This is because given any target location vector $S = (S_1, \ldots, S_d) \in [0, 1]^d$ and any estimated vector $\hat{S} = (\hat{S}_1, \ldots, \hat{S}_d) \in [0, 1]^d$, we have $\|S - \hat{S}\|_\infty \leq \|S - \hat{S}\|_2 \leq \sqrt{d} \|S - \hat{S}\|_\infty$.

An extension of Theorem 3 holds when the number of targets $k$ increases with the number of queries $n$ at rate $k = \Theta(n^\nu)$ with $\nu \in (0, 0.5)$. This is because our non-asymptotic bounds in Theorems 1 and 2 hold for any integer values of $k$ and $n$. The only difference between the proof of this result and that of Theorem 3 is that we need to choose $\gamma = (1 + \alpha)n^{\nu} \log 2$ for some $\alpha > 0$ in both directions and let $\beta = n^{-\nu - 0.5}$ in the converse proof.

**D. Numerical Simulation**

We numerically simulate the case of $k = 2$ and $d = 1$. Consider the measurement-dependent BSC with parameter $\zeta = 0.4$ and various Lipschitz continuous functions $f(\cdot)$. We assume that the target random variables $(S_1, S_2)$ are independently and uniformly distributed over the set $[0, 1]$. In Figure 2, the simulated achievable resolution of our non-adaptive query procedure in Algorithm 1 is plotted and compared to the theoretical predictions in Theorem 3.

In our simulation, for each $n \in \mathbb{N}$, the target resolution is chosen to be the reciprocal of $M$ satisfying

$$\log M = \frac{nC(2) + \sqrt{nV(2, \varepsilon)\Phi^{-1}(\varepsilon)} - \frac{1}{2} \log n}{2},$$  \hspace{1cm} (22)$$

For each number of queries $n$, the non-adaptive procedure in Algorithm 1 is run independently $10^4$ times and the achievable resolution is then calculated. From Figure 2, we observe that Theorem 3 provides tight approximation to the non-asymptotic performance of Algorithm 1 under various choices of the Lipschitz continuous functions.
Note that the function \( \Gamma(s) \).

For subsequent analysis, given any \( M \).

**Detailed Description of the Non-adaptive Query Procedure**

Consider any integer \( M \in \mathbb{N} \). Partition the unit cube \([0, 1]^d\) into \( M^d \) equal-sized disjoint cubes \( S_1, \ldots, S_{M^d} \) and consider any \( M^d \) binary vectors \( x = \{ x^n(1), \ldots, x^n(M^d) \} \). For each \( l \in [n] \), the \( l \)-th query is designed as

\[
A_l := \bigcup_{j \in [M^d]: x_l(j) = 1} S_i,
\]

(25)

For subsequent analysis, given any \( s \in [0, 1] \), define the following quantization function

\[
q(s) := \lfloor s M \rfloor,
\]

(26)

Suppose we have \( k \) targets with location vectors \( s^k := (s_1, \ldots, s_k) \), where for each \( i \in [k] \), the \( d \)-dimensional location vector of the \( i \)-th target \( s_i \) is given by \((s_{i,1}, \ldots, s_{i,d})\). For each \( i \in [k] \) and \( j \in [d] \), let \( w_{i,j}(s^k) = q(s_{i,j}) \) denote the quantized value of \( s_{i,j} \) and let \( w_i(s^k) = (w_{i,1}(s^k), \ldots, w_{i,d}(s^k)) \) denote the quantized location of \( s_i \). Recall the definition of \( \Gamma(\cdot) \) in (24).

We number\(^2\) the sub-cubes so that for each \( i \in [k] \), the \( i \)-th target with location vector \( s_i \) lies in the sub-cube \( S_l(\Gamma(w_i(s^k))) \).

\(^2\)This can be done as follows. We can partition the unit cubes into equal-sized sub-cubes \( \{B_{i_1, \ldots, i_d}\}_{(i_1, \ldots, i_d) \in [M^d]} \). The partition is done by partitioning each dimension into equal-length intervals and then taking the formed sub-cubes as the partition where index \( i_j \) denotes the index of the partition in the \( j \)-th dimension for each \( j \in [d] \). Then our proposed partition is formed by choosing \( S_i = B_{\Gamma^{-1}(i)} \) for each \( i = [M^d] \).
It is possible that two targets can be quantized into the same partition, i.e., there exists a pair of distinct indices \((i, j) \in [k]^2\) such that \(w_i = w_j\). If so, the detected number of targets would be smaller than \(k\). To account for these cases, we define the set of unique quantized locations as

\[
\mathcal{W}_p(s^k) := \{ w \in [M^d] : \exists i \in [k] \text{ s.t. } \Gamma(w_i(s^k)) = w \}. \tag{27}
\]

and define the number of distinct quantized targets as

\[
k_p(s^k) := |\mathcal{W}_p(s^k)|. \tag{28}
\]

Furthermore, let \(w_p^1(s^k)\) be the collection of elements in \(\mathcal{W}_p(s^k)\) in an increasing order and let \(w_p^i(s^k, i)\) be the \(i\)-th element. From the definition, \(w_p^i(s^k) \in \mathcal{L}(k, M^d)\) (cf. (12)). For each \(l \in [n]\), the noiseless answer to the query \(A_l\) (cf. (25)) is

\[
z_l := 1\{ \exists i \in [k] : s_i \in A_l \} \tag{29}
\]

\[
= 1\{ \exists i \in [k] : s_i \in \bigcup_{j : x_i(j) = 1} S_j \} \tag{30}
\]

\[
= 1\{ \exists i \in [k_p(s^k)] : x_i(w_p^i(s^k, i)) = 1 \}. \tag{31}
\]

The noisy response \(Y_l\) to the query \(A_l\) is obtained by passing the noiseless response \(z_l\) over the measurement-dependent channel \(P_{Y_l|z}\).

Recall the definitions of \(\mathcal{L}(k, M)\) in (12) and \(D^{n,t}(\gamma)\) in (11). Given noisy responses \(Y^n = (Y_1, \ldots, Y_n)\), if there exists an unique tuple \((i_1, \ldots, i_k) \in \mathcal{L}(k, M^d)\) such that \((x^n(i_1), \ldots, x^n(i_k), Y^n) \in D^{n,t}(\gamma)\), the decoder \(g\) first produces estimates of quantized values \((W_1, \ldots, W_k)\) where for each \(i \in [k]\), \(W_i = (W_{i,1}, \ldots, W_{i,d}) = \Gamma^{-1}(i_k)\). Subsequently, the decoder \(g\) outputs the set of estimated location vectors \(S_m\) as the centers of sub-cubes \(S_1, \ldots, S_k\), i.e., \(S_m = \{\hat{S}_1, \ldots, \hat{S}_k\}\) with

\[
\hat{S}_{i,j} = \frac{2W_{i,j} - 1}{2M}, \tag{32}
\]

where \(\hat{S}_{i,j}\) is the \(j\)-th element of the \(d\)-dimensional vector \(\hat{S}_i\) for each \(i \in [k]\) and \(j \in [d]\); if more than one such tuple exists, then randomly pick one; if no such tuple exists, then the decoder \(g\) sets \(t = k - 1\) and uses \(t\) in the role of \(k\) to continue the previous steps until \(t = 0\), as described in Algorithm 1.

C. Analysis of Excess-Resolution Events

Given the above non-adaptive query procedure (see also Algorithm 1) using binary query vectors \(x = (x^n(1), \ldots, x^n(M^d))\), an error (excess-resolution event) for location vectors \(s^k = (s_1, \ldots, s_k)\) occurs if one of the following three events occurs

- \(E_1(s^k, x, Y^n)\): \(x^n(w_p^1(s^k), 1), \ldots, x^n(w_p^k(s^k, k_p(s^k)), Y^n) \notin D^{n,k_p(s^k)}(\gamma)\);
- \(E_2(s^k, x, Y^n)\): there exists a tuple \((i_1, \ldots, i_{k_p(s^k)}) \in \mathcal{L}(k_p(s^k), M^d)\) such that \((x^n(i_1), \ldots, x^n(i_{k_p(s^k)}), Y^n) \notin D^{n,k_p(s^k)}(\gamma)\) and \((i_1, \ldots, i_{k_p(s^k)}) \neq w_p^1(s^k)\);
- \(E_3(s^k, x, Y^n)\): for some \(t \in [k_p(s^k) + 1 : k]\), there exists a tuple \((i_1, \ldots, i_t) \in \mathcal{L}(t, M^d)\) such that \((x^n(i_1), \ldots, x^n(i_t), Y^n) \notin D^{n,t}(\gamma)\).

Thus, using the non-adaptive query procedure in Algorithm 1, given any target location vectors \(s^k = (s_1, \ldots, s_k) \in [0, 1]^{dk}\), any binary query vectors \(x = (x^n(1), \ldots, x^n(M^d))\) and any \(M \in \mathbb{N}\), the excess-resolution probability with respect to resolution level \(\frac{1}{M}\) satisfies

\[
P_e(s^k, x^n) := \max \left\{ \Pr \left\{ \exists \ i \in [k] : \min_{\hat{S} \in S_m} |\hat{S} - s_i|_\infty > \frac{1}{M} \right\}, \Pr \left\{ \exists \ \hat{S} \in S_m : \min_{i \in [k]} ||\hat{S} - s_i||_\infty > \frac{1}{M} \right\} \right\} \tag{33}
\]

\[
\leq \Pr \left\{ \bigcup_{i \in [3]} E_i(s^k, x, Y^n) \right\} \tag{34}
\]

\[
\leq \sum_{i \in [3]} \Pr \{ E_i(s^k, x, Y^n) \}, \tag{35}
\]

where the probability term is calculated with respect to the conditional distribution of the noisy responses \(Y^n\) given \(x\) induced by the measurement-dependent channel \(P_{Y^n|X^n}\).
D. Employ Random Query Vectors and Change of Measure

For subsequent analysis, we will employ random coding ideas for which we consider random binary vectors \( X := \{ X^n(1), \ldots, X^n(M^d) \} \), where for each \( i \in [M^d] \), the length-\( n \) vector \( X^n(i) \) is generated i.i.d. from \( P_X \), a Bernoulli distribution with parameter \( p \in (0, 1) \).

The joint distribution of \((X,Y^n)\) under the non-adaptive query procedure with queries \( A^n = (A_1, \ldots, A_n) \) is

\[
P^A_{X^n,Y^n}(x,y^n) = \left( \prod_{j \in [M^d]} P^n_{X}(x^n(j)) \right) \times \left( \prod_{i \in [n]} P^A_{Y^n|X}(y_i| \exists i \in [k_p(s^k)] : X_i(w^+_p(s^k, i)) = 1) \right).
\]  

(36)

Unless otherwise stated, the probability terms are calculated according to the joint distribution \( P^n_{X^n,Y^n} \). For any \( \eta \in \mathbb{R}_+ \), define the following typical set of sequences:

\[
T^n(M,d,p,\eta) := \left\{ x = \{ x^n(j) \}_{j \in [M^d]} \in [0,1]^{M^d} : \max_{i \in [n]} |g_{i,x}^M,M,n(x) - p| \leq \eta \right\},
\]

(37)

where

\[
q^M_{i,x}(x) := \frac{1}{M^d} \sum_{j \in [M^d]} x_i(j).
\]

(38)

Similarly to the proof of [4, Theorem 1], we have that for any \( \eta > 0 \),

\[
\mathbb{E}[P_e(s^k,X)] \leq \mathbb{E}[P_e(s^k,X) 1\{ X \in T^n(M,d,p,\eta) \}] + \mathbb{P}(X \notin T^n(M,d,p,\eta)) \leq \mathbb{E}[P_e(s^k,X) 1\{ X \in T^n(M,d,p,\eta) \}] + 4n \exp(-2M^d\eta^2),
\]

(39)

where (40) follows from [22, Lemma 22], which upper bounds the atypicality probability of i.i.d. random variables using the union bound and Hoeffding’s inequality.

In the rest of this section, we focus on upper bounding the first term in (40). Recall the definition of the measurement-independent distribution \( P^I_{X^n}(x^n) \) in (41).

\[
P^I_{X^n,Y^n}(x,y^n) = \left( \prod_{j \in [M^d]} P^n_{X}(x^n(j)) \right) \times \left( \prod_{i \in [n]} P^I_{Y^n|X}(y_i| \exists i \in [k_p(s^k)] : X_i(w^+_p(s^k, i)) = 1) \right).
\]

(41)

Using the assumption on the measurement-dependent channel in (2), for any \( x \in T^n(M,d,p,\eta) \), we have

\[
\log \frac{P^A_{X^n,Y^n}(x,y^n)}{P^I_{X^n,Y^n}(x,y^n)} = \sum_{i \in [n]} \log \frac{P^I_{y^n|X}(y_i| \exists i \in [k_p(s^k)] : X_i(w^+_p(s^k, i)) = 1)}{P^I_{y^n|X}(y_i| \exists i \in [k_p(s^k)] : X_i(w^+_p(s^k, i)) = 1)} \leq nK \eta c(f(p)).
\]

(42)

Combining (35) and (42), we have

\[
\mathbb{E}[P_e(s^k,X) 1\{ X \in T^n(M,d,p,\eta) \}] \\
\leq \mathbb{P}(\bigcup_{i \in [n]} E_1(s^k,X,Y^n) 1\{ X \in T^n(M,d,p,\eta) \}) \\
\leq \exp(n\eta \mu c(f(p))) \mathbb{P}(\bigcup_{i \in [n]} E_1(s^k,X,Y^n)) \\
\leq \exp(n\eta \mu c(f(p))) \left( \mathbb{P}(E_1(s^k,X,Y^n)) + \mathbb{P}(E_2(s^k,X,Y^n)) + \mathbb{P}(E_3(s^k,X,Y^n)) \right).
\]

(43)

(44)

E. Further Upper Bound on the Excess-Resolution Probability

From the definition of excess-resolution events in Section IV-C, the first term in the bracket of (44) is

\[
\mathbb{P}(E_1(s^k,X,Y^n)) = \mathbb{P}(\bigcup_{i \in [n]} \{(X^n(w^+_p(s^k,1)), \ldots, X^n(w^+_p(s^k,k_p(s^k)), Y^n) \notin D, k_p(s^k)\}(\gamma) \}).
\]

(45)

We will then bound the second and the third terms inside the bracket of (44).

For subsequent analysis, define the following set for each \( j \in [k_p(s^k)] \):

\[
I_j(s^k) := \left\{ (i_1, \ldots, i_{k_p(s^k)}) \in \mathcal{L}(k_p(s^k), M^d) : (i_1, \ldots, i_{k_p(s^k)}) \neq w^+_p(s^k) \right\}
\]

and \[ \left\{ l \in [k_p(s^k)] : i_l \neq w^+_p(s^k, l) \right\} = j \].

(46)

Note that \( I_j(s^k) \) denotes the set of length-\( k_p(s^k) \) vectors that are different from the quantized location vectors \( w^+_p(s^k) \) by exactly \( j \) elements. For any \( i := (i_1, \ldots, i_{k_p(s^k)}) \in I_j(s^k) \), we use \( X^n_i \) to denote the random binary vectors \((X^n(i_1), \ldots, X^n(i_{k_p(s^k)}))\) and the joint distribution of \((X^n(i_i), Y^n)\) are induced by the joint distribution \( P^I_{X^n,Y^n} \) in (41). We use \( \mathcal{J}(i,s^k) \) to denote the
where \( (47) \) follows from the definition of the joint distribution \( P_{X,Y}^{(p)} \) in (41) and \( P_{X,Y}^{(p),k} \) is the induced marginal distribution of the joint distribution \( P_{X,Y}^{(p),k} \) in (6) for any \( J \subseteq [k] \).

Using the information spectrum method introduced [18], [23], we can upper bound the probability of \( \Pr \{ \mathcal{E}_2(s^k, X, Y^n) \} \) as follows:

\[
\begin{align*}
\Pr \{ \mathcal{E}_2(s^k, X, Y^n) \} &:= \sum_{j \in [k]} \sum_{(x^n, y^n) \in \mathcal{I}_j(s^k)} \Pr \{ (X^n(i), Y^n) : (X^n(i), Y^n) \in \mathcal{I}_j(s^k, (s^k)(\gamma)) \} \\
&\leq \sum_{j \in [k]} \sum_{(x^n, y^n) \in \mathcal{I}_j(s^k)} \Pr \{ (X^n(i), Y^n) \in \mathcal{I}_j(s^k, (s^k)(\gamma)) \} \\
&= \sum_{j \in [k]} \sum_{(x^n, y^n) \in \mathcal{I}_j(s^k)} \mathcal{I}_j(s^k, (s^k)(\gamma)) \prod_{t \in [k]} P_{X,Y}^{(p),k} \left( y^n | x^n_{j(i)} \right) \\
&\leq \sum_{j \in [k]} \sum_{(x^n, y^n) \in \mathcal{I}_j(s^k)} \exp(-\gamma) \prod_{t \in [k]} P_{X,Y}^{(p),k} \left( y^n | x^n_{j(i)} \right) \\
&= \sum_{j \in [k]} \sum_{i_1, \ldots, i_{k+1} \in [k]} \exp(-\gamma) \\
&\leq \frac{2^k \gamma}{M} \\
&\leq \frac{2^k \gamma}{M}.
\end{align*}
\]

where (48) follows since from the definition of \( \mathcal{I}_j(s^k) \) in (46), \( \cup_{j \in [k]} \mathcal{I}_j(s^k) = \mathcal{L}(k, s^k, M) \setminus \{ w_j(k) \} \), (52) follows from the definition of \( D_{n,t}(\cdot) \) in (11), (54) follows since the size of \( \mathcal{I}_j(s^k) \) is no greater than \( \binom{k}{j} M^d \) and (56) follows since \( k \leq k \).

Similarly to steps leading to (54), we can show that:

\[
\begin{align*}
\Pr \{ \mathcal{E}_3(s^k, X, Y^n) \} &\leq \sum_{t \in [k+1]} 2^t \exp(-\gamma) \\
&\leq 2^k \sum_{t \in [k+1]} \exp(-\gamma) \\
&\leq 2^k \left( k - k \gamma(s^k) \right) \exp(-\gamma) \\
&\leq k2^k \exp(-\gamma).
\end{align*}
\]

Combining (40), (44), (56) and (60), we conclude that for any \( s^k = (s_1, \ldots, s_k) \) and any \( M \in \mathbb{N} \), using the above non-adaptive query procedure, the excess-resolution probability satisfies:

\[
E[P_e(s^k, X)] \leq 4n \exp(-2M^d \eta) + \exp(nm\eta(q(p))) \left( (k+1)2^k \exp(-\gamma) + \Pr \{ \mathcal{E}_1(s^k, X, Y^n) \} \right).
\]
where for each \( \hat{P} \) from the set of estimated location vectors \( \mathcal{W} \), targets are quantized into the same region, i.e., there exists a \( \tilde{\epsilon} \) such that for any \( \hat{P} \) \( \tilde{\epsilon} \). Furthermore, let \( \hat{P} \) \( \tilde{\epsilon} \) generated independently from a distribution \( \mathcal{X}\mathcal{Y} \). Our subsequent analysis focuses on the probability term in (61). Without loss of generality, we assume that \( k_p(s^k) = t \) for some \( t \in [k] \) and let \( w^t_p(s^k) = [1 : t] \). Recall the definitions of the (conditional) mutual information density in (7) and its statistics in (8) to (23). Then we have

\[
\begin{align*}
\Pr_{P^{(p)}_{X|Y^n}} \{ \mathcal{E}_1(s^k, X, Y^n) \} &= \Pr_{P^{(p)}_{X|Y^n}} \{ (X^n_{[t]}, Y^n) \notin \mathcal{D}^{n,t}(\gamma) \} \\
&= \Pr_{P^{(p)}_{X|Y^n}} \{ (X^n_{[t]}, Y^n) \notin \bigcap_{\mathcal{J}\subset[t]} \mathcal{D}^{n,t}(\gamma) \} \\
&\leq \max_{t\in[k]} \Pr_{P^{(p)}_{X|Y^n}} \{ (X^n_{[t]}, Y^n) \notin \bigcap_{\mathcal{J}\subset[t]} \mathcal{D}^{n,t}(\gamma) \}. 
\end{align*}
\]

(62)

(63)

Therefore, combining (61) and (63) leads to

\[
\mathbb{E}[\Pr_e(S^k, X)] \leq 4n \exp(-2M^d\eta) + \exp(n\eta\epsilon(c(p)))((k + 1)2^k \exp(-\gamma) + \max_{t\in[k]} \Pr_{P^{(p)}_{X|Y^n}} \{ (X^n_{[t]}, Y^n) \notin \bigcap_{\mathcal{J}\subset[t]} \mathcal{D}^{n,t}(\gamma) \}),
\]

(64)

where (64) follows from the definition of the joint distributions of \( I^{(p),k}_{X^k|Y^n} \) in (6) and \( I^{(p)}_{X|Y^n} \) in (41).

Finally, the existence of deterministic binary vectors \( x = \{x^1(1), \ldots, x^d(M^d)\} \) with desired performance is guaranteed by the simple fact that \( \mathbb{E}[X] \leq a \) implies that there exists \( x \leq a \) for any random variable \( X \) and real number \( a \).

V. PROOF OF THE NON-ASYMPTOTIC CONVERSE BOUND (THEOREM 2)

A. Lower Bound the Excess-Resolution Probability

Consider any non-adaptive query procedure with queries \( A^n \subseteq [0,1]^nd \) and the decoder \( g : Y^n \rightarrow S_m \subseteq [0,1]^{dk} \) such that the worst-case excess-resolution probability with respect to \( \delta \in \mathbb{R}_+ \) is upper bounded by \( \epsilon \in (0,1) \), i.e.,

\[
P_e(n, k, d, \delta) = \sup_{f_S \in \mathcal{F}([0,1]^d)} \max \left\{ \Pr \left\{ \exists i \in [k] : \min_{S \in S_m} |\hat{S} - S_i|_\infty > \delta \right\}, \Pr \left\{ \exists \hat{S} \in S_m : \min_{i \in [k]} |\hat{S} - S_i|_\infty > \delta \right\} \right\} \leq \epsilon,
\]

(65)

where for each \( (i, j) \), \( S_{i,j} \) is the \( j \)-th element of the \( i \)-th target location vector \( S_i \) and \( \hat{S}_j \) is the \( j \)-th element of a vector \( \hat{S} \) from the set of estimated location vectors \( S_m \).

In the rest of the proof, without loss of generality, we consider the case where the location vector \( S_i \) of each target is generated independently from a uniform distribution \( f^d_j \) over the unit cube of dimension \( d \), i.e., \( [0,1]^d \). For any \( \beta \leq \frac{\epsilon}{2} \leq 0.5 \), let \( \tilde{M} := \left\lfloor \frac{\beta}{\epsilon} \right\rfloor \). Similar to (26), define the quantization function \( q_\beta(\cdot) \):

\[
q_\beta(s) = \lfloor s \tilde{M} \rfloor
\]

(66)

Given queries \( A^n \subseteq [0,1]^dn \), the noiseless responses \( Z^n \) is a sequence of independent random variables where for each \( t \in [n] \), \( Z_t = \mathbb{1}(\exists i \in [k] : S_i = (S_{i,1}, \ldots, S_{i,d}) \in A_t) \) is a Bernoulli random variable with parameter \( 1 - (1 - (1 - (1 - (1 - \epsilon)) / \epsilon)) \). For simplicity, for each \( (i,j) \in [k] \times [d] \), let \( W_{i,j}(S^k) := q_\beta(S_{i,j}) \) and let \( W_i(S^k) \) denote \( W_{i,1}, \ldots, W_{i,d} \). Similarly, we use \( W(\hat{S}) \) to the vector \( (q_\beta(S_{1,1}), \ldots, q_\beta(S_{d,1})) \) for any \( \hat{S} \in S_m \). Furthermore, recall the definition of \( \Gamma(\cdot) \) in (24) and define the sets

\[
\mathcal{W}(S^k) := \{ \Gamma(W_1(S^k)), \ldots, \Gamma(W_d(S^k)) \},
\]

(67)

\[
\mathcal{W}(S_m) := \{ \Gamma(W(\hat{S})) : \hat{S} \in S_m \},
\]

(68)

Furthermore, let \( \tilde{k}_p(S^k) := |\mathcal{W}(S^k)| \) denote the size of the set \( \mathcal{W}(S^k) \). Note that \( \tilde{k}_p(S^k) \) can be smaller than \( k \) if two or more targets are quantized into the same region, i.e., there exists \( (i,j) \in [k]^2 \) such that \( i \neq j \) and \( W_{i}(S^k) = W_{j}(S^k) \). The two sets \( \mathcal{W}(S^k) \) and \( \mathcal{W}(S_m) \) are said to be equal if both sets contain the same elements, i.e., \( \mathcal{W}(S^k) \cap \mathcal{W}(S_m) \cap \mathcal{W}(S_m)^c = \emptyset \).
We then have

\[
\Pr\{\hat{W}(S_m) \neq W(S^k), |\hat{W}(S_m)| = |W(S^k)|\} = \Pr\Big\{\hat{W}(S_m) \neq W(S^k), |\hat{W}(S_m)| = |W(S^k)|, \quad \exists \ i \in [k]: \min_{S \in S_m} |\hat{S} - S_i| > \delta \Big\} \\
+ \Pr\Big\{\hat{W}(S_m) \neq W(S^k), |\hat{W}(S_m)| = |W(S^k)|, \quad \forall \ i \in [k]: \min_{S \in S_m} |\hat{S} - S_i| \leq \delta \Big\}
\]

\[
\leq \Pr\Big\{\exists \ i \in [k]: \min_{S \in S_m} |\hat{S} - S_i| > \delta \Big\} \\
+ \Pr\Big\{\hat{W}(S_m) \neq W(S^k), |\hat{W}(S_m)| = |W(S^k)|, \quad \forall \ i \in [k]: \min_{S \in S_m} |\hat{S} - S_i| \leq \delta \Big\}
\]

\[
\leq \varepsilon + \sum_{i \in [k]} \Pr\Big\{W_i(S^k) \notin \hat{W}(S_m) \text{ and } \min_{S \in S_m} |\hat{S} - S_i| \leq \delta \Big\} \\
= \varepsilon + \sum_{i \in [k]} \Pr\Big\{W_i(S^k) \notin \hat{W}(S_m) \text{ and } \exists \hat{S} \in S_m: |\hat{S} - S_i| \leq \delta \Big\} \\
\leq \varepsilon + \sum_{i \in [k]} \Pr\Big\{\exists \hat{S} \in S_m: W_i(S^k) \neq \hat{W}(\hat{S}) \text{ and } |\hat{S} - S_i| \leq \delta \Big\} \\
\leq \varepsilon + \sum_{i \in [k]} \sum_{S \in S_m} 2d\delta M \\
\leq \varepsilon + 2k|S_m|d\beta \\
\leq \varepsilon + 2k^2d\beta,
\]

where (71) follows from (65), (72) follows since \(\hat{W}(S_m) \neq W(S^k)\) and \(|\hat{W}(S_m)| = |W(S^k)|\) imply that there exists \(i \in [k]\) such that \(W_i(S^k) \notin \hat{W}(S_m)\), (76) follows similarly to [4, Eq. (89)], and (77) follows from the definition of \(M\) and (78) follows since the size of \(S_m\) is upper bounded by the number of targets \(k\).

**B. Connection with A Multiple Access Channel**

Note that the location vector of each target is assumed independent and for each \(i \in [k], S_i = (S_{i,1}, \ldots, S_{i,d})\) is uniformly distributed over the unit cube of dimension \(d\), each quantized location \(W_{i,j}\) is uniformly distributed over the message set \([M]\) and thus \(\Gamma(W_i)\) is uniformly distributed over \([M^d]\). Thus, without loss of generality, we can write \(W(S^k) = (W_1, \ldots, W_{k_p(S^k)})\), where each message is generated independently from a uniform distribution over the message set \([M]\). Furthermore, the noiseless responses \(Z^\circ\) can be understood as the outputs of a “OR” type multiple access channel of \(k_p(S^k)\) channel inputs. Specifically, if we let \(X := (X^\circ(1), \ldots, X^\circ(k_p(S^k)))\) be a collection \(k_p(S^k)\) independent random vectors where for each \(i \in [k_p(S^k)], X^\circ(i)\) is independently and non-identically distributed according to the Bernoulli product distributions \(P_{X^\circ}^{A,\circ}\), where

\[
P_{X^\circ}^{A,\circ}(x) = \prod_{i \in [n]} (|A_i|^{x_i}(1 - |A_i|)^{1-x_i}).
\]

Then \(Z^\circ = (Z_1, \ldots, Z_n)\) is given by

\[
Z_t = \mathbb{I}(\exists \ i \in [k_p(S^k)] : X_t(i) = 1), \quad \forall \ t \in [n].
\]

The noisy responses \(Y^\circ\) is then the output of passing \(Z^\circ\) over measurement-dependent channels \(P_{Y^\circ|Z^\circ}^{A,\circ}\). Then, under the condition that \(|\hat{W}(S_m)| = |W(S^k)|\), the set \(S_m\) can then be understood as estimates \((\hat{W}_1, \ldots, \hat{W}_{k_p(S^k)})\) of the messages \(W(S^k) = (W_1, \ldots, W_{k_p(S^k)})\). Therefore, the error event \(\hat{W}(S_m) \neq W(S^k)\) and \(|\hat{W}(S_m)| = |W(S^k)|\) is equivalent to \((\hat{W}_1, \ldots, \hat{W}_{k_p(S^k)}) \neq (W_1, \ldots, W_{k_p(S^k)})\), which is the error probability of decoding messages over a measurement-dependent noisy channel \(P_{Y^\circ|Z^\circ}^{A,\circ}\) with channel input \(Z^\circ\) and output \(Y^\circ\). Note that the channel \(P_{Y^\circ|Z^\circ}^{A,\circ}\) is in fact a special case of the multiple access channel with channel inputs \(X := (X^\circ(1), \ldots, X^\circ(k_p(S^k)))\), each of which correspond to a message \(W_i\).
Therefore, (78) implies that the excess-resolution probability of a non-adaptive query procedure with queries \( A^n \) is lower bounded by the error probability of decoding \( \hat{k}_p(S^k) \) messages over the above mentioned multiple access channel and a small penalty term, i.e.,

\[
\varepsilon \geq \Pr \left\{ \mathcal{W}(S_m) \neq \mathcal{W}(S^k) \text{ and } |\mathcal{W}(S_m)| = |\mathcal{W}(S^k)| \right\} - 2k^2d\beta,
\]

(81)

\[
= \Pr \left\{ (\hat{W}_1, \ldots, \hat{W}_{\hat{k}_p(S^k)}) \neq (W_1, \ldots, W_{\hat{k}_p(S^k)}) \right\} - 2k^2d\beta.
\]

(82)

Note that the probability term in (82) depends on the queries \( A^n \) through the measurement-dependent noisy channel.

**C. Lower Bound the Minimal Achievable Resolution Using Information Spectrum Method**

We then lower bound the first term in (82) using the information spectrum method introduced in [23, Lemma 4].

Note that for any \( s^k = (s_1, \ldots, s_k) \in \{0, 1\}^{k,t} \), the number of present targets \( \hat{k}_p(s^k) \) with respect to the quantization level \( \hat{M} \) takes values in \([k]\). Without loss of generality, we first consider \( s_k \) such that \( \hat{k}_p(s^k) = t \) for some \( t \in [k] \). Thus, the probability term in (82) is written as the error probability of a \( t \)-user multiple access channel, i.e.,

\[
\Pr \left\{ (\hat{W}_1, \ldots, \hat{W}_{\hat{k}_p(S^k)}) \neq (W_1, \ldots, W_{\hat{k}_p(S^k)}) | S^k : \hat{k}_p(S^k) = t \right\} = \Pr \left\{ (\hat{W}_1, \ldots, \hat{W}_t) \neq (W_1, \ldots, W_t) \right\},
\]

where the multiple access channel model \( P_{Y^n|X^n}^{A^n,t} \) results from the definitions of \( Z^n \) in (80) and the measurement-dependent channel \( P_{Y|X}^{f(p),t} \) from (6), i.e.,

\[
P_{Y^n|X^n}^{A^n,t}(y^n|x^n(1), \ldots, x^n(t)) = \prod_{l \in [n]} P_{Y|X}^{f(|A_l|)}(y|1 \{ \exists j \in [t] : x_l(j) = 1 \}),
\]

(84)

where we use \( X^n_{[l]} \) to denote \( X^n(1), \ldots, X^n(t) \) and use \( x^n_{[l]} \) similarly.

Furthermore, let \( P_{X^n|Y^n}^{A^n} \) denote the following joint distribution

\[
P_{X^n|Y^n}^{A^n}(x^n_{[l]}, y^n) = \prod_{i \in [t]} (P_{X}^{A_i}(x^n(i))) P_{Y^n|X^n_{[t]}}^{A^n,t}(y^n|x^n_{[t]}),
\]

where \( P_{X}^{A_i} \) denotes the Bernoulli distribution with \( P_{X}^{A_i}(1) = |A_i| \) and \( X^n(i) \) is generated i.i.d. from \( P_{X}^{A_i} \). For any \( l \in [n] \) and \( J \subset [t] \), let \( P_{X^n|Y^n}^{A_i|t} \) and \( P_{Y^n|X^n_{[J]}} \) be induced by \( P_{X^n|Y^n}^{A^n|t} \).

Similar to (7) and (10), for any query \( A \subseteq [0, 1]^m \) any \( J \subset [t] \) and any \( l \in N \), define the following (conditional) mutual information density

\[
\gamma_{J_{[l]}}^{A,l}(x_{[l]}; y) := \log \frac{P_{Y^n|X^n_{[l]}}^{A,i}(y|x_{[l]})}{P_{Y^n|X^n_{l}}(y|x_{[J]})},
\]

and for any sequence of queries \( A^n \)

\[
D_{\gamma_{J_{[t]}}}^{A^n}(\gamma) := \left\{ (x^n_{[t]}, y^n) \in \{0, 1\}^{tn} \times Y^n : \sum_{l \in [n]} \gamma_{J_{[l]}}^{A,l}(x_{[l]}; y) > d(t - |J|) \log \hat{M} - \gamma \right\}, \ J \subset [k],
\]

(87)

Similar to [23, Lemma 4], when queries \( A^n \) are used, we have

\[
\Pr \left\{ (\hat{W}_1, \ldots, \hat{W}_t) \neq (W_1, \ldots, W_t) \right\} \geq \Pr_{P_{X^n|Y^n}^{A^n,t}} \left\{ (X^n, Y^n) \notin \bigcup_{\mathcal{J} \subset [t]} D_{\gamma_{J_{[t]}}}^{A^n}(\gamma) \right\} - \sum_{\mathcal{J} \subset [t]} \exp(-\gamma)
\]

(88)

\[
= \Pr_{P_{X^n|Y^n}^{A^n,t}} \left\{ (X^n, Y^n) \notin \bigcup_{\mathcal{J} \subset [t]} (D_{\gamma_{J_{[t]}}}^{A^n}(\gamma))^c \right\} - 2^t \exp(-\gamma)
\]

(89)

\[
\geq \Pr_{P_{X^n|Y^n}^{A^n,t}} \left\{ (X^n, Y^n) \in (D_{\gamma_{J_{[t]}}}^{A^n}(\gamma))^c \right\} - 2^k \exp(-\gamma)
\]

(90)

\[
= \Pr_{P_{X^n|Y^n}^{A^n,t}} \left\{ \sum_{l \in [n]} 1_{|A_l|}(x_{[l]}; Y_l) \leq dt \log \hat{M} - \gamma \right\} - 2^k \exp(-\gamma).
\]

(91)

Combining (82), (83) and (89), we have that for any non-adaptive query procedure with queries \( A^n \) satisfying (65),

\[
\varepsilon \geq \max_{t \in [k]} \Pr_{P_{X^n|Y^n}^{A^n,t}} \left\{ \sum_{l \in [n]} 1_{|A_l|}(x_{[l]}; Y_l) \leq dt \log \hat{M} - \gamma \right\} - 2^k \exp(-\gamma) - 2k^2d\beta.
\]

(92)
Recalling that  \( n = \left\lfloor \frac{\beta}{2} \right\rfloor \) and noting that (92) holds for any queries \( \mathcal{A}^n \), we have

\[
- \log \delta(n, k, d, \varepsilon) \leq \sup_{\mathcal{A}^n \in [0, 1]^n} \left\{ \psi \in \mathbb{R}_+ : \max_{t \in [k]} \Pr_{X^n_{i[t]} Y^n} \left\{ \sum_{t \in [n]} |a_t| t (X_{i[t]}; Y_t) \leq dt \psi + dt \log \beta - \gamma \right\} \leq \varepsilon + 2^k \exp(-\gamma) + 2^{2k} d \beta \right\}.
\]

(93)

VI. PROOF OF SECOND-ORDER ASYMPTOTICS (THEOREM 3)

In this section, we present the proof of second-order asymptotic result in Theorem 3, which essentially applies the Berry-Esseen theorem to the non-asymptotic bounds in Theorems 1 and 2 respectively.

A. Achievability Proof

We start with the non-asymptotic achievability bound in (64). The probability terms are calculated with respect to \( (P^{f(p,t)}_{X^n_i Y^n})^n \) unless otherwise stated. Note that for each \( t \in [k] \),

\[
\Pr \left\{ (X^n_{i[t]}, Y^n) \notin \bigcap_{\mathcal{J} \subset [t]} \mathcal{D}^{n,t}_{\mathcal{J}}(\gamma) \right\} = \Pr \left\{ (X^n_{i[t]}, Y^n) \in \bigcup_{\mathcal{J} \subset [t]} \left( \mathcal{D}^{n,t}_{\mathcal{J}}(\gamma) \right)^c \right\}
\]

(94)

\[
\leq \sum_{\mathcal{J} \subset [t]} \Pr \left\{ (X^n_{i[t]}, Y^n) \in \left( \mathcal{D}^{n,t}_{\mathcal{J}}(\gamma) \right)^c \right\}
\]

(95)

\[
= \sum_{\mathcal{J} \subset [t]} \Pr \left\{ \sum_{t \in [n]} f^{(p),t}_{\mathcal{J}}(X_{i,t}; Y_t) \leq (t - |\mathcal{J}|) d \log M + \gamma \right\}
\]

(96)

Choose \( M \) and \( \gamma \) such that for some \( \varepsilon \in (0, 1) \),

\[
\gamma = \frac{1}{2} \log n
\]

(97)

\[
d \log M = \min_{t \in [k]} nC_{\phi}(p, t) + \sqrt{nV_{\phi}(p, t)} \Phi^{-1}(\varepsilon) - \frac{1}{2} \log n.
\]

(98)

Thus, for any \( t \in [k] \), we have

\[
dt \log M \leq nC_{\phi}(p, t) + \sqrt{nV_{\phi}(p, t)} \Phi^{-1}(\varepsilon) - \frac{1}{2} \log n.
\]

(99)

When \( \mathcal{J} = \emptyset \), the Berry-Esseen theorem [20], [21] implies that

\[
\Pr \left\{ \sum_{t \in [n]} f^{(p),t}_{\emptyset}(X_{i,t}; Y_t) \leq dt \log M + \gamma \right\}
\]

\[
\leq \Pr \left\{ \sum_{t \in [n]} f^{(p),t}_{\emptyset}(X_{i,t}; Y_t) \leq nC_{\phi}(p, t) + \sqrt{nV_{\phi}(p, t)} \Phi^{-1}(\varepsilon) - \frac{1}{2} \log n + \gamma \right\}
\]

(100)

\[
\leq \varepsilon + \frac{6T_{\phi}(p, t)}{\sqrt{nV_{\phi}(p, t)}}.
\]

(101)

Note that for each \( l \in [n] \), the measurement-dependent channel \( P^{f(p),k_{\phi}((s^k))}_{Y^n|X^n_{i[t]} X_{i[t]+1}} \) in (47) satisfies the assumptions for [13, Lemmas 1 and 2]. For any \( t \in [2:k] \) and \( \mathcal{J} \subset [t] \) such that \( \mathcal{J} \neq \emptyset \), similarly to [13, Lemmas 1 and 2], we have

\[
\frac{C_{\mathcal{J}}(p, t)}{|\mathcal{J}|} > \frac{C_{\emptyset}(p, t)}{t} > \frac{C_{\emptyset}(p, k)}{k},
\]

(102)

where (102) follows from standard calculation by noting that i) from the definition of \( C_{\mathcal{J}}(p, t) \) in (8),

\[
C_{\mathcal{J}}(p, t) = I(X_{i[t]}, Y; X_{\mathcal{J}}),
\]

(103)

with \( \mathcal{J}^c = [t] \setminus \mathcal{J} \) and ii) the joint distribution of \( (X_1, \ldots, X_t, Y) \) in (6) implies that \( X_i \) and \( X_j \) are not conditionally independent given \( Y \). For simplicity, we let

\[
\kappa(t) := \min_{\mathcal{J} \subset [t]: \mathcal{J} \neq \emptyset} \left( \frac{C_{\mathcal{J}}(p, t)}{|\mathcal{J}|} - \frac{C_{\emptyset}(p, t)}{t} \right).
\]

(104)
From (102), \( \kappa(t) > 0 \). Thus, for any \( \varepsilon \in (0, 1) \) and any \( \mathcal{J} \subset [t] \) such that \( \mathcal{J} \neq \emptyset \), there exists a positive constant \( \omega(t) \) such that

\[
\Pr \left\{ \sum_{t \in [n]} t_{J}^f(p,t)(X_{t;[t]}; Y_{t}) \leq |\mathcal{J}| + \log M + \gamma \right\} \\
= \Pr \left\{ \sum_{t \in [n]} t_{J}^f(p,t)(X_{t;[t]}; Y_{t}) \leq \frac{|\mathcal{J}|(n\gamma(p,t) + \sqrt{nV\gamma(p,t)\Phi^{-1}(\varepsilon)} - \frac{1}{2} \log n)}{t} + \gamma \right\} \\
\leq \Pr \left\{ \sum_{t \in [n]} t_{J}^f(p,t)(X_{t;[t]}; Y_{t}) \leq \kappa(C(p,t) - \kappa(t)) + \sqrt{nV\gamma(p,t)\Phi^{-1}(\varepsilon)} \right\} \\
\leq \exp(-n\omega(t)),
\]

where (105) follows from (99), (106) follows since \( 1 \leq |\mathcal{J}| < t \) and (104) implies that

\[
\frac{|\mathcal{J}|C\gamma(p,t)}{t} \leq C_{\mathcal{J}}(p,t) - |\mathcal{J}|\kappa(t) \leq C_{\mathcal{J}}(p,t) - \kappa(t),
\]

and (107) follows from the Chernoff bound that establishes the exponential convergence of the probability term in (106) with the positive real number \( \omega(t) \) being the exponent.

Combining (64), (96), (101) and (107), for \( M \) in (98) and \( \gamma \) in (97), we conclude

\[
\mathbb{E}[P_{\gamma}(S^k, X)] \\
\leq 4n \exp(-2M^d\eta) + \exp(n\eta\epsilon f(p))) \left( \frac{k + 1}{2n} + \varepsilon + \max_{t \in [k]} \left( \frac{6T\gamma(p,t)}{\sqrt{nV\gamma(p,t)^3}} + 2^t \exp(-n\omega) \right) \right)
\]

\[
\leq 4n \exp(-2M^d\eta) + \exp(n\eta\epsilon f(p))) \left( \frac{k + 1}{2n} + \varepsilon + \max_{t \in [k]} \left( \frac{6T\gamma(p,t)}{\sqrt{nV\gamma(p,t)^3}} + 2^t \exp(-n\omega) \right) \right),
\]

The result in (110) implies that there exists binary vectors \( \tilde{x} = (\tilde{x}^1(1), \ldots, \tilde{x}^n(M^d)) \) such that the excess-resolution probability with respect to the resolution level \( \frac{1}{k} \) for any location vectors \( s^k = (s_1, \ldots, s_k) \) is upper bounded by the right hand side of (110) with \( M \) chosen in (98).

We shall show that the right hand side of (110) is upper bounded by \( \varepsilon \) with proper choice of \( \eta \) when \( n \) is large. Let

\[
\eta := \sqrt{\frac{d \log M}{2M^d}}.
\]

Similarly to [4, Eq. (104)-(106)], we can then verify that as \( n \to \infty \), \( 4n \exp(-2M^d\eta^2) \to 0 \) and \( \exp(n\eta\epsilon f(p))) \to 1 \). Furthermore, since \( \mathcal{A} \) and \( \mathcal{Y} \) are both finite, from [15, Lemma 47], we conclude that \( V\gamma(p,t) \) and \( T\gamma(p,t) \) are both finite for any \( t \in [k] \). Therefore, for \( n \) sufficiently large, the right hand side of (110) is essentially \( \varepsilon \).

Note that the above result holds for any \( p \in (0, 1) \). Thus, for any \( n \in \mathbb{N} \) and \( \varepsilon \in (0, 1) \), the minimal achievable resolution \( \delta^*(n, k, d, \varepsilon) \) satisfies that

\[
-\log \delta^*(n, k, d, \varepsilon) \geq \max_{p \in (0, 1)} \min_{t \in [k]} \frac{nC\gamma(p,t) + \sqrt{nV\gamma(p,t)\Phi^{-1}(\varepsilon)} - \frac{1}{2} \log n}{dt}.
\]

Finally, the achievability proof is completed by using the second inequality in (102) which states that \( \frac{C\gamma(p,t)}{t} \geq \frac{C\gamma(p,k)}{k} \) for all \( t \in [k] \), noting that \( \frac{C\gamma(p,t)}{t} \) is the dominant term in (112) for \( n \) sufficiently large and finally concluding that for any \( (k, d, \varepsilon) \in \mathbb{N}^2 \times (0, 1) \),

\[
-\log \delta^*(n, k, d, \varepsilon) \geq \max_{p \in (0, 1)} \frac{nC\gamma(p,k) + \sqrt{nV\gamma(p,k)\Phi^{-1}(\varepsilon)} - \frac{1}{2} \log n}{dk} \\
= \frac{C(k) + \sqrt{nV(k,\varepsilon)\Phi^{-1}(\varepsilon)} - \frac{1}{2} \log n}{dk},
\]

where (114) follows from the definitions of \( C(k) \) in (16) and \( V(k,\varepsilon) \) in (17).

\textbf{B. Converse Proof}

We start with the non-asymptotic converse in bound (93).
For any queries $A^n \in [0, 1]^n$ and any $t \in [k]$, let

$$C_{A^n}^t := \frac{1}{n} \sum_{i \in [n]} \mathbb{E} \left[ t_{0}^{[A]_t} (X_i; Y_t) \right].$$

(115)

$$V_{A^n}^t := \frac{1}{n} \sum_{i \in [n]} \text{Var} \left[ t_{0}^{[A]_t} (X_i; Y_t) \right].$$

(116)

$$T_{A^n}^t := \frac{1}{n} \sum_{i \in [n]} \mathbb{E} \left[ \left| t_{0}^{[A]_t} (X_i; Y_t) - \mathbb{E} [ t_{0}^{[A]_t} (X_i; Y_t) ] \right|^3 \right].$$

(117)

Choose $\gamma = \frac{1}{2} \log n$ and $\beta = \frac{1}{\sqrt{n}}$. Consider those $A^n$ such that there exists $V_\gamma > 0$ satisfying $V_\gamma \leq V_{A^n}^t$. The Berry-Esseen theorem implies that for any $A^n \in [0, 1]^n$, when $n$ is sufficiently large so that $\frac{2^k d + 2k^2}{d \sqrt{n}} < 1 - \varepsilon$,

$$\sup \left\{ \psi : \max_{t \in [k]} \mathbb{P} \left[ \sum_{i \in [n]} t_{0}^{[A]_t} (X_i; Y_t) \leq dt \psi + dt \log \beta - \gamma \right] \leq \varepsilon + 2^k \exp(-\gamma) + 2k^2 d \beta \right\}$$

$$\leq \min_{t \in [k]} \frac{nC_{A^n}^t + \sqrt{nV_{A^n}^t} \Phi^{-1} \left( \varepsilon + \frac{2^k d + 2k^2}{d \sqrt{n}} + \frac{6T_{A^n}^t}{\sqrt{nV_{A^n}^t}} \right) + \frac{dt + 1}{2} \log n}{dt}$$

(118)

$$\leq \frac{nC_{A^n}^t}{dk} + \frac{dt + 1}{2} \log n$$

(119)

$$R(\mathcal{A}, k, d),$$

(120)

where (119) follows because the dominant term in (118) is $\frac{nC_{A^n}^t}{dt}$ and the inequality $\frac{nC_{A^n}^t}{dk} \geq \frac{nC_{A^n}^t}{k}$ holds similarly to [13, Lemma 1] since for each $l \in [n]$, the measurement-dependent channel $P_{X^n_{[l]}|X_{[l]}}(y|1 \{ \exists j \in [l] : x_l(j) = 1 \})$ in (84) satisfies the assumptions for [13, Lemmas 1].

Then for those queries $A^n$ such that $V_{A^n}^t = 0$, we have $t_{0}^{[A]_t} (X_i; Y_t)$ is a constant for each $l \in [n]$ and thus

$$\sup \left\{ \psi : \max_{t \in [k]} \mathbb{P} \left[ \sum_{i \in [n]} t_{0}^{[A]_t} (X_i; Y_t) \leq dt \psi + dt \log \beta - \gamma \right] \leq \varepsilon + 2^k \exp(-\gamma) + 2k^2 d \beta \right\}$$

$$\leq \min_{t \in [k]} \frac{nC_{A^n}^t}{dt} + \frac{dt + 1}{2} \log n$$

(121)

$$\leq \frac{nC_{A^n}^t}{dk} + \log(n)$$

(122)

$$\leq R(\mathcal{A}, k, d),$$

(123)

where (123) follows similarly to (120) and using the fact that $V_{A^n}^t = 0$.

Recall the definitions of $C_{\mathcal{F}}(p, t)$ in (8), $C_0(k)$ in (16) and $V(k, \varepsilon)$ in (17) and the definition of $\mathcal{P}_k$ be the set of optimizers $p^*$ for the optimization problem in (16). Note that

$$\sup_{A^n} C_{A^n}^k \leq \sup_{A \in [0, 1]^d} C_0(|A|, k)$$

(124)

$$\leq \max_{p \in (0, 1)} C_0(p, k)$$

(125)

$$= C(k).$$

(126)

Because $C_{A^n}^t$ is the dominant term in $R(\mathcal{A}, k, d)$, for $n$ sufficiently large, we have that for any $p^* \in \mathcal{P}_k$,

$$\sup_{A^n} R(\mathcal{A}, k, d) = \sup_{A^n : \forall i \in [n], |A_i| = p^*} R(\mathcal{A}, k, d)$$

(127)

$$\leq \sup_{A^n : \forall i \in [n], |A_i| = p^*} \frac{nC_0(k) + \sqrt{nV(k, \varepsilon)} \Phi^{-1} \left( \varepsilon + \frac{2^k d + 2k^2}{d \sqrt{n}} + \frac{6T_{A^n}^t}{\sqrt{nV_{A^n}^t}} \right) + \frac{(dk + 1) \log n}{2}}{dk}$$

(128)

$$\leq \frac{nC_0(k) + \sqrt{nV(k, \varepsilon)} \Phi^{-1} \left( \varepsilon + \frac{(dk + 1) \log n}{2} + O(1) \right) + \log n + O(1)}{dk}$$

(129)

$$\leq \frac{nC_0(k) + \sqrt{nV(k, \varepsilon)} \Phi^{-1} \left( \varepsilon + \frac{(dk + 1) \log n}{2} + O(1) \right)}{dk}$$

(130)
where (129) follows from the Taylor expansion of $\Phi^{-1}(\cdot)$ and the fact that $T_{\mathcal{A}_n}^{k,n}$ is finite.

Combining (93), (120) and (130), we have that for $n$ sufficiently large, the minimal achievable resolution satisfies

$$
-\log \delta^*(n, k, d, \varepsilon) \leq \sup_{\mathcal{A}_n} R(A^n, k, d)
\leq nC_0(k) + \sqrt{nV(k, \varepsilon)}\Phi^{-1}(\varepsilon) + \log n + O(1),
$$

where the constant term $O(1)$ depends on the number of targets $k$.

VII. CONCLUSION

We derived the fundamental resolution limit of optimal non-adaptive query procedures for simultaneous search for multiple targets over the unit cube. Our proof used the information spectrum method for a multiple access channel [18], the finite blocklength information theory [15] and the inequalities for random access channel coding [13]. Our non-asymptotic and asymptotic results provided benchmarks and important insights for the design of non-adaptive query procedures.

For future work, it would be worthwhile to investigate low complexity practical non-adaptive query procedures that achieve the benchmarks derived in this paper. Furthermore, it would be of interest to study the fundamental limit of adaptive query procedures for any measurement-dependent discrete memoryless channel and quantify the benefit of adaptivity. In this line of research, one might modify the adaptive query procedures in [4] or explore the sorted posterior matching algorithm. Finally, one can also consider the current problem under privacy constraints [24], [25] and obtain the privacy-utility tradeoff [26], [27] for non-adaptive and adaptive query procedures.

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REFERENCES